

On backward parabolic Ito equations with periodic and mixed in time conditions

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August 29, 2012

Abstract

We study linear backward stochastic partial differential equations of parabolic type with special boundary conditions in time. The standard Cauchy condition at the terminal time is replaced by a condition that holds almost surely and mixes the random values of the solution at different times, including the terminal time, initial time and continuously distributed times. Uniqueness, solvability and regularity results for the solutions are obtained. In particular, conditions of existence of periodic in time and "almost periodic" solutions are obtained for backward SPDEs.

AMS 1991 subject classification: Primary 60J55, 60J60, 60H10. Secondary 34F05, 34G10.

Key words and phrases: stochastic partial differential equations, backward SPDEs, parabolic Ito equations, non-local boundary conditions, non-local in time conditions, mixed in time conditions.

1 Introduction

Partial differential equations and stochastic partial differential equations (SPDEs) have fundamental significance for natural sciences, and various boundary value problems for them were widely studied. Usually, well-posedness of a boundary value depends on the choice of the boundary value conditions. For the deterministic parabolic equations, well-posedness requires the correct choice of the initial condition. For example, consider the heat equation $u'_t = u''_{xx}$, $t \in [0, T]$. For this equation, a boundary value problem with the Cauchy condition at initial time $t = 0$ is well-posed, and a boundary value problem with the Cauchy condition at terminal time $t = T$ is ill-posed. It is known also that the problems for deterministic parabolic equation are well-posed for periodic type condition $u(x, 0) = u(x, T)$ (see, e.g., Dokuchaev (1994, 1995), Fife (1964), Hess

(1991), Lieberman (1999), Nakao (1984), Shelukhin (1993), Vejvoda (1982)). Less is known for parabolic equation with more general non-local in time conditions and for SPDEs.

Boundary value problems for SPDEs are well studied in the existing literature for the case of forward parabolic Ito equations with the Cauchy condition at initial time (see, e.g., Alós *et al* (1999), Bally *et al* (1994), Da Prato and Tubaro (1996), Gyöngy (1998), Krylov (1999), Maslowski (1995), Pardoux (1993), Rozovskii (1990), Walsh (1986), Zhou (1992), and the bibliography there). Many results have been also obtained for the backward parabolic Ito equations with Cauchy condition at terminal time, as well as for pairs of forward and backward equations with separate Cauchy conditions at initial time and the terminal time respectively (see, e.g., Yong and Zhou (1999), and the author's papers (1992), (2005),(2011), (2012)). Note that a backward SPDE cannot be transformed into a forward equation by a simple time change, unlike as for the case of deterministic equations. Usually, a backward SPDE is solvable in the sense that there exists a diffusion term being considered as a part of the solution that helps to ensure that the solution is adapted to the driving Brownian motions.

There are also results for SPDEs with boundary conditions that mix the solution at different times that may include initial time and terminal time. This category includes stationary type solutions for forward SPDEs (see, e.g., Caraballo *et al* (2004), Chojnowska-Michalik (19987), Chojnowska-Michalik and Goldys (1995), Duan *et al* (2003), Mattingly (1999) Mohammed *et al* (2008), Sinai (1996), and the references here). There are also results for periodic solutions of SPDEs (Chojnowska-Michalik (1990), Feng and Zhao (2012), Klünger (2001)). It is difficult to expect that, in general, a SPDE has a periodic in time solution $u(\cdot, t)|_{t \in [0, T]}$ in a usual sense of exact equality $u(\cdot, t) = u(\cdot, T)$ that holds almost surely. The periodicness of the solutions was usually considered in the sense of the distributions. In Feng and Zhao (2012), the periodicness was established for self-adjoint parabolic equation in a stronger sense as a "random periodic solution (see Definition 1.1 from Feng and Zhao (2012)); this definition does not assume the equality $u(\cdot, t) = u(\cdot, T)$ The results here were obtain for semi-linear equations with self-adjoint main operator.

The paper addresses these and related problems for linear parabolic type SPDEs that are not necessary self adjoint. We consider the Dirichlet condition at the boundary of the state domain. Following Dokuchaev (2008), standard boundary value Cauchy condition at the initial time is replaces by a condition that mixes in one equation the values of the solution at different times over given time interval, including the terminal time and continuously distributed times. In Dokuchaev (2008), related problems were considered for forward parabolic type SPDEs with non-local boundary conditions on expectations such that non-random initial values and the ex-

pectation of a functional of the path were included. For instance, an expectation of a random terminal value could be required to match the non-random initial value. In other words, the setting from Dokuchaev (2008) excluded non-local boundary value conditions represented as equations for random values that hold almost surely. In the present paper, we were able to include such boundary value conditions; it appears that it can be done with the replacement of forward SPDEs for backward SPDEs (Theorems 3.1-3.5). This setting allowed to the condition of periodicness and some other non-local boundary value conditions that has to be satisfied almost surely. This is a novel setting comparing with the conditions for the expectations considered in Dokuchaev (2008)), or with periodic conditions for the distributions, or with conditions in Klünger (2001) and Feng and Zhao (2012). These conditions include, for instance, conditions $\kappa u(\cdot, 0) = u(\cdot, T)$ a.e. with $\kappa \in [-1, 1]$ (Theorem 3.5). For deterministic parabolic equations, similar conditions were introduced in Dokuchaev (1994) (see also Theorem 2.2 from Dokuchaev (2004)). Less restrictive sufficient conditions of existence of "almost periodic" solutions of backward SPDEs are obtained in Corollary 3.1.

Uniqueness, existence, and regularity results for the solutions are obtained in L_2 -setting.

2 The problem setting and definitions

We are given a standard complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a right-continuous filtration \mathcal{F}_t of complete σ -algebras of events, $t \geq 0$. We are given also a N -dimensional Wiener process $w(t)$ with independent components; it is a Wiener process with respect to \mathcal{F}_t .

Assume that we are given an open domain $D \subset \mathbf{R}^n$ such that either $D = \mathbf{R}^n$ or D is bounded with C^2 -smooth boundary ∂D . Let $T > 0$ be given, and let $Q \triangleq D \times [0, T]$.

We will study the following boundary value problem in Q

$$d_t u + (\mathcal{A}u + \varphi) dt + \sum_{i=1}^N B_i \chi_i dt = \sum_{i=1}^N \chi_i(t) dw_i(t), \quad t \geq 0, \quad (2.1)$$

$$u(x, t, \omega)|_{x \in \partial D} = 0 \quad (2.2)$$

$$u(\cdot, T) - \Gamma u(\cdot) = \xi. \quad (2.3)$$

Here $u = u(x, t, \omega)$, $\varphi = \varphi(x, t, \omega)$, $\chi_i = \chi_i(x, t, \omega)$, $(x, t) \in Q$, $\omega \in \Omega$.

In (2.3), Γ is a linear operator that maps functions defined on $Q \times \Omega$ to functions defined on $D \times \Omega$. For instance, the case where $\Gamma u = u(\cdot, 0)$ is not excluded; this case corresponds to the periodic type boundary condition

$$u(\cdot, T) - u(\cdot, 0) = \xi. \quad (2.4)$$

In (2.1),

$$\mathcal{A}v \triangleq \sum_{i=1}^n \frac{\partial}{\partial x_i} \sum_{j=1}^n \left(b_{ij}(x, t, \omega) \frac{\partial v}{\partial x_j}(x) \right) + \sum_{i=1}^n f_i(x, t, \omega) \frac{\partial v}{\partial x_i}(x) + \lambda(x, t, \omega) v(x), \quad (2.5)$$

where b_{ij}, f_i, x_i are the components of b, f , and x respectively, and

$$B_i v \triangleq \frac{dv}{dx}(x) \beta_i(x, t, \omega) + \bar{\beta}_i(x, t, \omega) v(x), \quad i = 1, \dots, N. \quad (2.6)$$

We assume that the functions $b(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}^{n \times n}$, $\beta_j(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}^n$, $\bar{\beta}_i(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$, $f(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}^n$, $\lambda(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$, $\chi_i(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$, and $\varphi(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$ are progressively measurable with respect to \mathcal{F}_t for all $x \in \mathbf{R}^n$, and the function $\xi(x, \omega) : \mathbf{R}^n \times \Omega \rightarrow \mathbf{R}$ is \mathcal{F}_0 -measurable for all $x \in \mathbf{R}^n$. In fact, we will also consider φ and ξ from wider classes. In particular, we will consider generalized functions φ .

If the functions b, f, λ, φ , and ξ , are deterministic, then $\chi_i \equiv 0$ and equation (2.1) is deterministic.

Spaces and classes of functions

We denote by $\|\cdot\|_X$ the norm in a linear normed space X , and $(\cdot, \cdot)_X$ denote the scalar product in a Hilbert space X .

We introduce some spaces of real valued functions.

Let $G \subset \mathbf{R}^k$ be an open domain, then $W_q^m(G)$ denote the Sobolev space of functions that belong to $L_q(G)$ together with the distributional derivatives up to the m th order, $q \geq 1$.

We denote by $|\cdot|$ the Euclidean norm in \mathbf{R}^k , and \bar{G} denote the closure of a region $G \subset \mathbf{R}^k$.

Let $H^0 \triangleq L_2(D)$, and let $H^1 \triangleq W_2^1(D)$ be the closure in the $W_2^1(D)$ -norm of the set of all smooth functions $u : D \rightarrow \mathbf{R}$ such that $u|_{\partial D} \equiv 0$. Let $H^2 = W_2^2(D) \cap H^1$ be the space equipped with the norm of $W_2^2(D)$. The spaces H^k and $W_2^k(D)$ are called Sobolev spaces, they are Hilbert spaces, and H^k is a closed subspace of $W_2^k(D)$, $k = 1, 2$.

Let H^{-1} be the dual space to H^1 , with the norm $\|\cdot\|_{H^{-1}}$ such that if $u \in H^0$ then $\|u\|_{H^{-1}}$ is the supremum of $(u, v)_{H^0}$ over all $v \in H^1$ such that $\|v\|_{H^1} \leq 1$. H^{-1} is a Hilbert space.

We shall write $(u, v)_{H^0}$ for $u \in H^{-1}$ and $v \in H^1$, meaning the obvious extension of the bilinear form from $u \in H^0$ and $v \in H^1$.

We denote by $\bar{\ell}_k$ the Lebesgue measure in \mathbf{R}^k , and we denote by $\bar{\mathcal{B}}_k$ the σ -algebra of Lebesgue sets in \mathbf{R}^k .

We denote by $\bar{\mathcal{P}}$ the completion (with respect to the measure $\bar{\ell}_1 \times \mathbf{P}$) of the σ -algebra of subsets of $[0, T] \times \Omega$, generated by functions that are progressively measurable with respect to \mathcal{F}_t .

We introduce the spaces

$$\begin{aligned} X^k(s, t) &\triangleq L^2([s, t] \times \Omega, \bar{\mathcal{P}}, \bar{\ell}_1 \times \mathbf{P}; H^k), \\ Z_t^k &\triangleq L^2(\Omega, \mathcal{F}_t, \mathbf{P}; H^k), \\ \mathcal{C}^k(s, t) &\triangleq C([s, T]; Z_T^k), \quad k = -1, 0, 1, 2. \end{aligned}$$

The spaces $X^k(s, t)$ and $Z_t^k(s, t)$ are Hilbert spaces.

In addition, we introduce the spaces

$$Y^k(s, t) \triangleq X^k(s, t) \cap \mathcal{C}^{k-1}(s, t), \quad k = 1, 2,$$

with the norm $\|u\|_{Y^k(s, T)} \triangleq \|u\|_{X^k(s, t)} + \|u\|_{\mathcal{C}^{k-1}(s, t)}$.

For brevity, we shall use the notations $X^k \triangleq X^k(0, T)$, $\mathcal{C}^k \triangleq \mathcal{C}^k(0, T)$, and $Y^k \triangleq Y^k(0, T)$.

Conditions for the coefficients

To proceed further, we assume that Conditions 2.1-2.3 remain in force throughout this paper.

Condition 2.1 *The matrix $b = b^\top$ is symmetric and bounded. In addition, there exists a constant $\delta > 0$ such that*

$$y^\top b(x, t, \omega) y - \frac{1}{2} \sum_{i=1}^N |y^\top \beta_i(x, t, \omega)|^2 \geq \delta |y|^2 \quad \forall y \in \mathbf{R}^n, (x, t) \in D \times [0, T], \omega \in \Omega. \quad (2.7)$$

Condition 2.2 *The functions $f(x, t, \omega)$, $\lambda(x, t, \omega)$, $\beta_i(x, t, \omega)$, and $\bar{\beta}_i(x, t, \omega)$, are bounded.*

Condition 2.3 *The mapping $\Gamma : Y^1 \rightarrow Z_T^0$ is linear and continuous.*

Condition 2.3 allows, for instance, to consider Γ such $\Gamma u = u(\cdot, 0)$, i.e., it covers periodic boundary value conditions (2.4). Another example includes the case where there exists an integer $m \geq 0$, a set $\{t_i\}_{i=1}^m \subset [0, T]$, and linear continuous operators $\tilde{\Gamma}_0 : L_2([0, T]; \mathcal{B}_1, \ell_1, H^0) \rightarrow H^0$, $\tilde{\Gamma}_i : H^0 \rightarrow H^0$, $i = 1, \dots, m$, such that

$$\Gamma u = \tilde{\Gamma}_0 u + \sum_{i=1}^m \tilde{\Gamma}_i u(\cdot, t_i).$$

In particular, it includes

$$\tilde{\Gamma}_0 u = \int_0^T k_0(t) u(\cdot, t) dt, \quad \tilde{\Gamma}_i u(\cdot, t_i) = k_i u(\cdot, t_i),$$

where $k_0(\cdot) \in L_2(0, T)$ and $k_i \in \mathbf{R}$. It covers also Γ such that

$$\tilde{\Gamma}_0 u = \int_0^T dt \int_D k_0(x, y, t) u(y, t) dx, \quad \tilde{\Gamma}_i u(\cdot, t_i)(x) = \int_D k_i(x, y) u(y, t_i) dy,$$

where $k_i(\cdot)$ are some regular enough kernels.

We introduce the set of parameters

$$\mathcal{P} \triangleq \left(n, D, T, \Gamma, \delta, \right. \\ \left. \text{ess sup}_{x, t, \omega, i} \left[|b(x, t, \omega)| + |f(x, t, \omega)| + |\lambda(x, t, \omega)| + |\beta_i(x, t, \omega)| + |\bar{\beta}_i(x, t, \omega)| \right] \right).$$

Sometimes we shall omit ω .

The definition of solution

Proposition 2.1 *Let $\zeta \in X^0$, let a sequence $\{\zeta_k\}_{k=1}^{+\infty} \subset L^\infty([0, T] \times \Omega, \ell_1 \times \mathbf{P}; C(D))$ be such that all $\zeta_k(\cdot, t, \omega)$ are progressively measurable with respect to \mathcal{F}_t , and let $\|\zeta - \zeta_k\|_{X^0} \rightarrow 0$. Let $t \in [0, T]$ and $j \in \{1, \dots, N\}$ be given. Then the sequence of the integrals $\int_0^t \zeta_k(x, s, \omega) dw_j(s)$ converges in Z_t^0 as $k \rightarrow \infty$, and its limit depends on ζ , but does not depend on $\{\zeta_k\}$.*

Proof follows from completeness of X^0 and from the equality

$$\mathbf{E} \int_0^t \|\zeta_k(\cdot, s, \omega) - \zeta_m(\cdot, s, \omega)\|_{H^0}^2 ds = \int_D dx \mathbf{E} \left(\int_0^t (\zeta_k(x, s, \omega) - \zeta_m(x, s, \omega)) dw_j(s) \right)^2.$$

Definition 2.1 Let $\zeta \in X^0$, $t \in [0, T]$, $j \in \{1, \dots, N\}$, then we define $\int_0^t \zeta(x, s, \omega) dw_j(s)$ as the limit in Z_t^0 as $k \rightarrow \infty$ of a sequence $\int_0^t \zeta_k(x, s, \omega) dw_j(s)$, where the sequence $\{\zeta_k\}$ is such as in Proposition 2.1.

Definition 2.2 Let $u \in Y^1$, $\chi_i \in X^0$, $i = 1, \dots, N$, and $\varphi \in X^{-1}$. We say that equations (2.1)-(2.2) are satisfied if

$$u(\cdot, t, \omega) = u(\cdot, T, \omega) + \int_t^T (\mathcal{A}u(\cdot, s, \omega) + \varphi(\cdot, s, \omega)) ds \\ + \sum_{i=1}^N \int_t^T B_i \chi_i(\cdot, s, \omega) ds - \sum_{i=1}^N \int_t^T \chi_i(\cdot, s) dw_i(s)$$

for all r, t such that $0 \leq r < t \leq T$, and this equality is satisfied as an equality in Z_T^{-1} .

Note that the condition on ∂D is satisfied in the sense that $u(\cdot, t, \omega) \in H^1$ for a.e. t, ω . Further, $u \in Y^1$, and the value of $u(\cdot, t, \omega)$ is uniquely defined in Z_T^0 given t , by the definitions of the corresponding spaces. The integrals with dw_i in (2.8) are defined as elements of Z_T^0 . The integral with ds in (2.8) is defined as an element of Z_T^{-1} . In fact, Definition 2.2 requires for (2.1) that this integral must be equal to an element of Z_T^0 in the sense of equality in Z_T^{-1} .

3 The main results

Theorem 3.1 *There exist a number $\kappa = \kappa(\mathcal{P}) > 0$ such that problem (2.1)-(2.3) has an unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^1 \times (X^0)^N$, for any $\varphi \in X^{-1}$, $\xi \in Z_T^0$, and any Γ such that $\|\Gamma\| \leq \kappa$, where $\|\Gamma\|$ is the norms of the operator $\Gamma : Y^1 \rightarrow Z_T^0$. In addition,*

$$\|u\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C \left(\|\varphi\|_{X^{-1}} + \|\xi\|_{Z_T^0} \right), \quad (3.1)$$

where $C = C(\kappa, \mathcal{P}) > 0$ is a constant that depends only on κ and \mathcal{P} .

Let \mathbb{I} denote the indicator function.

Theorem 3.2 *Let $\bar{\Gamma}_0$ in Condition 2.3 be such that there exists $\tau > 0$ such that $\bar{\Gamma}_0 u = \bar{\Gamma}_0(\mathbb{I}_{\{t \geq \tau\}} u)$. Then*

$$\|u\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C \left(\|\varphi\|_{X^{-1}} + \|u\|_{X^{-1}} + \|\xi\|_{Z_T^0} \right) \quad (3.2)$$

for all solutions $(u, \chi_1, \dots, \chi_N)$ of problem (2.1)-(2.3) in the class $Y^1 \times (X^0)^N$, where $C = C(\mathcal{P}) > 0$ depends only on \mathcal{P} and does not depend on u , φ and ξ .

Starting from now and up to the end of this section, we assume that Condition 3.1 holds.

Condition 3.1 (i) *The domain D is bounded.*

(ii) *The functions $b(x, t, \omega)$, $f(x, t, \omega)$, $\lambda(x, t, \omega)$, $\beta_i(x, t, \omega)$ and $\bar{\beta}_i(x, t, \omega)$ are differentiable in x for a.e. t, ω , and the corresponding derivatives are bounded.*

(iii) *$\beta_i(x, t, \omega) = 0$ for $x \in \partial D$, $i = 1, \dots, N$.*

(iv) *\mathcal{F}_0 is the \mathbf{P} -augmentation of the set $\{\emptyset, \Omega\}$.*

(v) There exists an integer $m \geq 0$, a set $\{t_i\}_{i=1}^m \subset [0, T]$, and linear continuous operators $\bar{\Gamma} : L_2(Q) \rightarrow H^0$, $\bar{\Gamma}_i : H^0 \rightarrow H^0$, $i = 0, 1, \dots, N$, such that the operators $\bar{\Gamma} : L_2([0, T]; \mathcal{B}_1, \ell_1, H^1) \rightarrow W_2^1(D)$ and $\bar{\Gamma}_i : H^1 \rightarrow W_2^1(D)$ are continuous and

$$\Gamma u = \bar{\Gamma}_0 u(\cdot, 0) + \mathbf{E}\{\bar{\Gamma} u + \sum_{i=1}^m \bar{\Gamma}_i u(\cdot, t_i)\}.$$

In particular, it follows from this condition that there exist modifications of β_i such that the functions $\beta_i(x, t, \omega)$ are continuous in x for a.e. t, ω . We assume that β_i are such functions.

Note that the assumptions on Γ imposed in Condition 3.1 allows to consider $\Gamma u = u(\cdot, 0)$, i.e., the periodic type boundary conditions (2.4).

Theorem 3.3 Assume that problem (2.1)-(2.3) with $\varphi \equiv 0$, $\xi \equiv 0$, does not admit non-zero solutions for all solutions $(u, \chi_1, \dots, \chi_N)$ of in the class $Y^1 \times (X^0)^N$. Then problem (2.1)-(2.3) has a unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^1 \times (X^0)^N$, for any $\varphi \in X^{-1}$, and $\xi \in H^0$. In addition,

$$\|u\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C(\|\varphi\|_{X^{-1}} + \|\xi\|_{H^0}), \quad (3.3)$$

where $C = C(\mathcal{P}) > 0$ does not depend on φ and ξ .

Theorem 3.4 There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ such that $\varepsilon \neq 0$, problem (2.1)-(2.2) with the boundary value condition

$$u(\cdot, T) - (1 + \varepsilon)\Gamma u = \xi \quad (3.4)$$

has a unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^1 \times (X^0)^N$ for any $\varphi \in X^{-1}$ and $\xi \in H^0$. In addition,

$$\|u\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C(\|\varphi\|_{X^{-1}} + \|\xi\|_{H^0}), \quad (3.5)$$

where $C = C(\varepsilon, \mathcal{P}) > 0$ does not depend on φ and ξ .

Corollary 3.1 Let $\kappa \in R$ be given. Then there exists $\varepsilon_0 = \varepsilon_0(\mathcal{P}, \kappa) > 0$ such that, for any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ such that $\varepsilon \neq 0$, for any $\varphi \in X^{-1}$, there exists a unique solution $(u, \chi_1, \dots, \chi_N)$ of problem (2.1), (2.2) in the class $Y^1 \times (X^0)^N$, such that

$$u(\cdot, T) = \kappa(1 + \varepsilon)u(\cdot, 0). \quad (3.6)$$

If $\kappa = 1$, then one may say that backward SPDE (2.1)-(2.2) allows an "almost periodic" solutions in this sense.

It would be interesting to find an example where an exact periodic condition is satisfied (i.e., when $\varepsilon = 0$ is allowed in (3.6)). So far, we found the following example.

Theorem 3.5 *Let $\bar{\beta}_i \equiv 0$ and let the functions b, f and λ be such that the operator \mathcal{A} can be represented as*

$$\mathcal{A}v = \sum_{i,j=1}^n b_{ij}(x, t, \omega) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^n \widehat{f}_i(x, t, \omega) \frac{\partial v}{\partial x_i}(x) + \widehat{\lambda}(x, t, \omega)v(x),$$

where $\widehat{\lambda}(x, t) \leq 0$, and where \widehat{f}_i are bounded functions. Further, let $\Gamma u = \kappa u(\cdot, 0)$, where $\kappa \in [-1, 1]$. Then problem (2.1)-(2.2), (3.6) has a unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^1 \times (X^0)^N$ for any $\varphi \in X^{-1}$ and $\xi \in H^0$. In addition, (3.3) holds with $C = C(\mathcal{P}) > 0$ that does not depend on φ and ξ .

4 Proofs

Let $s \in (0, T]$, $\varphi \in X^{-1}$ and $\Phi \in Z_s^0$. Consider the problem

$$\begin{aligned} d_t u + (\mathcal{A}u + \varphi) dt + \sum_{i=1}^N B_i \chi_i(t) dt &= \sum_{i=1}^N \chi_i(t) dw_i(t), \quad t \leq s, \\ u(x, t, \omega)|_{x \in \partial D}, & \\ u(x, s, \omega) &= \Phi(x, \omega). \end{aligned} \tag{4.1}$$

The following lemma represents an analog of the so-called "the first energy inequality", or "the first fundamental inequality" known for deterministic parabolic equations (see, e.g., inequality (3.14) from Ladyzhenskaya (1985), Chapter III).

Lemma 4.1 *Assume that Conditions 2.1–2.3 are satisfied. Then problem (4.1) has a unique solution a unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^1 \times (X^0)^N$ for any $\varphi \in X^{-1}(0, s)$, $\Phi \in Z_s^0$, and*

$$\|u\|_{Y^1(0,s)} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C (\|\varphi\|_{X^{-1}(0,s)} + \|\Phi\|_{Z_s^0}), \tag{4.2}$$

where $C = C(\mathcal{P})$ does not depend on φ and ξ .

(See, e.g., Dokuchaev (1991) or Theorem 4.2 from Dokuchaev (2010)).

Note that the solution $u = u(\cdot, t)$ is continuous in t in $L_2(\Omega, \mathcal{F}, \mathbf{P}, H^0)$, since $Y^1(0, s) = X^1(0, s) \cap \mathcal{C}^0(0, s)$.

Introduce operators $L_s : X^{-1}(0, s) \rightarrow Y^1(0, s)$ and $\mathcal{L}_s : Z_s^0 \rightarrow Y^1(0, s)$, such that $u = L_s \varphi + \mathcal{L}_s \Phi$, where $(u, \chi_1, \dots, \chi_N)$ is the solution of problem (4.1) in the class $Y^2 \times (X^1)^N$. By Lemma 4.1, these linear operators are continuous.

Introduce operators $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$ and $\mathcal{T} : X^{-1} \rightarrow Z_T^0$ such that $\mathcal{Q}\Phi + \mathcal{T}\varphi = \Gamma u$, where u is the solution in Y^1 of problem (4.1) with $s = T$, $\varphi \in X^{-1}$, and $\Phi \in Z_T^0$. It is easy to see that these operators are linear and continuous.

Proof of Theorem 3.1. For brevity, we denote $u(\cdot, t) = u(x, t, \omega)$. Clearly, $u \in Y^1$ is the solution of problem (2.1)-(2.3), if

$$\begin{aligned} u &= L_T \varphi + \mathcal{L}_T u(\cdot, T), \\ u(\cdot, T) - \Gamma u &= \xi. \end{aligned}$$

Since $\Gamma u = \mathcal{Q}u(\cdot, T) + \mathcal{T}\varphi$, we have

$$u(\cdot, T) - \mathcal{Q}u(\cdot, T) - \mathcal{T}\varphi = \xi.$$

Clearly, $\|\mathcal{Q}\| \leq \|\Gamma\|\|\mathcal{L}_T\|$, where $\|\mathcal{Q}\|$, $\|\Gamma\|$, and $\|\mathcal{L}_T\|$, are the norms of the operators $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$, $\Gamma : Y^1 \rightarrow Z_T^0$, and $\mathcal{L}_0 : Z_T^0 \rightarrow Y^1$, respectively. Since the operator $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$ is continuous, the operator $(I - \mathcal{Q})^{-1} : Z_T^0 \rightarrow Z_T^0$ is continuous for small enough $\|\mathcal{Q}\|$, i.e. for a small enough $\kappa > 0$. Hence

$$u(\cdot, T) = (I - \mathcal{Q})^{-1}(\xi + \mathcal{T}\varphi),$$

and

$$\begin{aligned} u &= L_T \varphi + \mathcal{L}_T u(\cdot, T) \\ &= L_T \varphi + \mathcal{L}_T (I - \mathcal{Q})^{-1}(\xi + \mathcal{T}\varphi). \end{aligned} \tag{4.3}$$

Then the proof of Theorem 3.1 follows. \square

Proof of Theorem 3.2. For a real $q > 0$, set $u_q(x, t, \omega) \triangleq e^{q(T-t)}u(x, t, \omega)$. Then u_q is the solution of problem (2.1)-(2.3) with φ replaced by $\varphi + q \cdot u$, and with λ and Γ replaced by λ_q and $\tilde{\Gamma}_q$, where

$$\lambda_q \triangleq \lambda + q, \quad \tilde{\Gamma}_q u = \tilde{\Gamma}_{0q} u + \sum_{i=1}^m \tilde{\Gamma}_{iq} u(\cdot, t_i),$$

with $\tilde{\Gamma}_{0q} u = \tilde{\Gamma}_0(e^{-q(T-t)}u)$, $\tilde{\Gamma}_{iq} u(\cdot, t_i) = \tilde{\Gamma}_i(e^{-q(T-t_i)}u(\cdot, t_i))$. By the assumptions on $\tilde{\Gamma}_0$ and by the choice of $t_i < T$, we have that $\|\tilde{\Gamma}_q\| \rightarrow 0$ as $q \rightarrow +\infty$, for the norm of the operator $\tilde{\Gamma}_q : Y^1 \rightarrow Z_T^T$. By Lemma 4.1 and Theorem 3.1, it follows that, for a large enough $q > 0$,

$$\begin{aligned} \|u_q\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} &\leq C_1 (\|\varphi + qu_q\|_{X^{-1}} + \|\xi\|_{H_0}) \\ &\leq C_2 (\|\varphi\|_{X^{-1}} + \|u_q\|_{X^{-1}} + \|\xi\|_{H_0}), \end{aligned}$$

where $C_1 = C_1(\mathcal{P}) > 0$ and $C_2 = C_2(q, \mathcal{P}) > 0$ do not depend on u, φ, ξ . Then the proof of Theorem 3.2 follows. \square

Starting from now, we assume that Condition 3.1 is satisfied, in addition to Conditions 2.1-2.3.

The following lemma represents an analog of the so-called "the second energy inequality", or "the second fundamental inequality" known for the deterministic parabolic equations (see, e.g., inequality (4.56) from Ladyzhenskaya (1985), Chapter III).

Lemma 4.2 *Problem (4.1) has a unique solution $(u, \chi_1, \dots, \chi_N)$ in the class $Y^2 \times (X^1)^N$ for any $\varphi \in X^0$, $\Phi \in Z_T^1$, and*

$$\|u\|_{Y^2} + \sum_{i=1}^N \|\chi_i\|_{X^1} \leq C \left(\|\varphi\|_{X^0} + \|\Phi\|_{Z_T^1} \right), \quad (4.4)$$

where $C > 0$ does not depend on φ and Φ ; it depends on \mathcal{P} and on the supremums of the derivatives listed in Condition 3.1(ii).

The lemma above represents a reformulation of Theorem 3.4 from Dokuchaev (2010) or Theorem 4.3 from Dokuchaev (2012). In the cited paper, this result was obtained under some strengthened version of Condition 2.1; this was restrictive. In Du and Tang (2012), this result was obtained without this restriction, i.e., under Condition 2.1 only.

Lemma 4.3 *The operator $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$ is compact.*

Proof of Lemma 4.3. Let $u = \mathcal{L}_0 \Phi$, where $\Phi \in H^0 = Z_0^0$. By the semi-group property of backward SPDEs from Theorem 6.1 from Dokuchaev (2011), we obtain that $u|_{t \in [0, s]} = \mathcal{L}_s u(\cdot, s)$ for all $s \in (0, T]$. By Lemmas 4.1 and 4.2, we have for $\tau \in \{t_1, \dots, t_m\}$ that

$$\begin{aligned} \|\mathbf{E} \bar{\Gamma}_i u(\cdot, \tau)\|_{W_2^1(D)}^2 &\leq C_0 \|u(\cdot, \tau)\|_{Z_0^1}^2 \leq C_1 \inf_{t \in [\tau, T]} \|u(\cdot, t)\|_{Z_t^1}^2 \leq \frac{C_1}{T - \tau} \int_{\tau}^T \|u(\cdot, t)\|_{Z_t^1}^2 dt \\ &\leq \frac{C_2}{T - \tau} \|\Phi\|_{Z_T^0}^2 \end{aligned}$$

and

$$\|\mathbf{E} \bar{\Gamma}_0 u\|_{W_2^1(D)}^2 \leq C_3 \mathbf{E} \int_0^T \|u(\cdot, t)\|_{Z_t^1}^2 dt \leq C_4 \|\Phi\|_{Z_T^0}^2.$$

for constants $C_i > 0$ which do not depend on Φ . Hence the operator $\mathcal{Q} : Z_T^0 \rightarrow W_2^1(D)$ is continuous. Since the embedding of $W_2^1(D)$ to H^0 and in Z_T^0 is a compact operator, the proof of Lemma 4.3 follows. \square

Proof of Theorem 3.3. By the assumptions, the equation $\mathcal{Q}\Phi = \Phi$ has the only solution $\Phi = 0$ in H^0 . By Lemma 4.3 and by the Fredholm Theorem, the operator $(I - \mathcal{Q})^{-1} : H^0 \rightarrow H^0$ is continuous. Then the proof of Theorem 3.3 follows from representation (4.3). \square

Proof of Theorem 3.4. By Lemma 4.3 and by the Fredholm Theorem again, for any $\varepsilon_0 \in (0, 1)$, there exists a finite set $\Lambda \subset \mathbf{C}$ such that the operator $(\lambda I - \mathcal{Q})^{-1} : H^0 \rightarrow H^0$ is continuous for all $\lambda \in (1 - \varepsilon_0, 1 + \varepsilon_0) \setminus \Lambda$. Then the proof of Theorem 3.4 follows from representation (4.3) again. \square

Corollary 3.1 is a special case of Theorem 3.4 with $\Gamma u = u(\cdot, 0)$.

Proof of Theorem 3.5. Let functions $\tilde{\beta}_i : Q \times \Omega \rightarrow \mathbf{R}^n$, $i = 1, \dots, M$, be such that

$$2b(x, t, \omega) = \sum_{i=1}^N \beta_i(x, t, \omega) \beta_i(x, t, \omega)^\top + \sum_{j=1}^M \tilde{\beta}_j(x, t, \omega) \tilde{\beta}_j(x, t, \omega)^\top,$$

and $\tilde{\beta}_i$ has the similar properties as β_i . (Note that, by Condition 2.1, $2b > \sum_{i=1}^N \beta_i \beta_i^\top$).

Let $\tilde{w}(t) = (\tilde{w}_1(t), \dots, \tilde{w}_M(t))$ be a new Wiener process independent on $w(t)$. Let $a \in L_2(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^n)$ be a vector such that $a \in D$. We assume also that a is independent from $(w(t) - w(t_1), \hat{w}(t) - \hat{w}(t_1))$ for all $t > t_1 > s$. Let $s \in [0, T)$ be given. Consider the following Ito equation

$$\begin{aligned} dy(t) &= \hat{f}(y(t), t) dt + \sum_{i=1}^N \beta_i(y(t), t) dw_i(t) + \sum_{j=1}^M \tilde{\beta}_j(y(t), t) d\tilde{w}_j(t), \\ y(s) &= x. \end{aligned} \tag{4.5}$$

Here $\hat{f} : D \times [0, T] \times \Omega \rightarrow \mathbf{R}^n$ is a vector functions with the components \hat{f}_i .

Let $y(t) = y^{a,s}(t)$ be the solution of (4.5), and let $\tau^{a,s} \triangleq \inf\{t \geq s : y^{a,s}(t) \notin D\}$.

Lemma 4.4 *There exists $\nu \in (0, 1)$ such that $\nu = \nu(\mathcal{P})$ depends only on \mathcal{P} and $\mathbf{P}(\tau^{a,s} > s + T) \leq \nu$, for all $s \geq 0$, and for any random vector a such that $a \in D$ a.s. and a does not depend on $w(t) - w(r)$ for all $t > r > s$.*

Note that if the functions $f(x, t, \omega) = f(x)$ and $\beta(x, t, \omega) = \beta(x)$ are non-random and constant in t , then existence of $\nu \in (0, 1)$ such that $\mathbf{P}(\tau^{a,s} > s + T) \leq \nu$ ($\forall a, s$) is obvious.

Proof of Lemma 4.4. In this proof, we will follow the approach from Dokuchaev (2004), p.296. Let $\mu = (f, \beta, x, s)$.

Clearly, there exists a finite interval $D_1 \triangleq (d_1, d_2) \subset \mathbf{R}$ and a bounded domain $D_{n-1} \subset \mathbf{R}^{n-1}$ such that $D \subset D_1 \times D_{n-1}$.

For $(x, s) \in D \times [0, s)$, let $\tau_1^{x,s} \triangleq \inf\{t \geq s : y_1^{x,s}(t) \notin D_1\}$, where $y_1^{x,s}(t)$ is the first component of the vector $y^{x,s}(t) = (y_1^{x,s}(t), \dots, y_n^{x,s}(t))$. We have that

$$\mathbf{P}(\tau^{x,s} > s + T) \leq \mathbf{P}(\tau_1^{x,s} > s + T) = \mathbf{P}(y_1^{x,s}(t) \in D_1 \ \forall t \in [s, s + T]). \tag{4.6}$$

Let

$$M^\mu(t) \triangleq \sum_{k=1}^N \int_s^t h_i(y^{x,s}(r), r) dw_i(r) + \sum_{k=N+1}^{N+M} \int_s^t h_i(y^{x,s}(r), r) d\tilde{w}_i(r), \quad t \geq s,$$

where $h = (h_1, \dots, h_{N+M})$ is a vector row with values in $\mathbf{R}^{1 \times (N+m)}$ that represents the first row of the matrix

$$(\beta_1, \dots, \beta_N, \hat{\beta}_1, \dots, \hat{\beta}_M)$$

with the values in $\mathbf{R}^{n \times (N+M)}$.

Let $\hat{D}_1 \triangleq (d_1 + K_1, d_2 + K_2)$, where $K_1 \triangleq -d_2 - T \sup_{x,t,\omega} |f_1(x, t, \omega)|$, $K_2 \triangleq -d_1 + T \sup_{x,t} |f_1(x, t, \omega)|$. Clearly, \hat{D}_1 depends only on n, D , and c_f . It is easy to see that

$$\mathbf{P}(y_1^{x,s}(t) \in D_1 \quad \forall t \in [s, s+T]) \leq \mathbf{P}(M^\mu(t) \in \hat{D}_1 \quad \forall t \in [s, s+T]). \quad (4.7)$$

Further,

$$h(y^{x,s}(t), t)^\top h(y^{x,s}(t), t) = |h(y^{x,s}(t), t)|^2 \in [\delta, c_\beta], \quad (4.8)$$

where

$$\delta = \inf_{x,s,\omega, \xi \in \mathbf{R}^n: |\xi|=1} 2\xi^\top b(x, t, \omega)\xi, \quad c_\beta = \sup_{x,s,\omega, \xi \in \mathbf{R}^n: |\xi|=1} 2\xi^\top b(x, t, \omega)\xi.$$

Clearly, $M^\mu(t)$ is a martingale vanishing at s with quadratic variation process

$$[M^\mu]_t \triangleq \int_s^t |h(y^{x,s}(r), r)|^2 dr, \quad t \geq s.$$

Let $\theta^\mu(t) \triangleq \inf\{r \geq s : [M^\mu]_r > t - s\}$. Note that $\theta^\mu(s) = s$, and the function $\theta^\mu(t)$ is strictly increasing in $t > s$ given (x, s) . By Dambis–Dubins–Schwarz Theorem (see, e.g., Revuz and Yor (1999)), the process $B^\mu(t) \triangleq M(\theta^\mu(t))$ is a Brownian motion vanishing at s , i.e., $B^\mu(s) = 0$, and $M^\mu(t) = B^\mu(s + [M^\mu]_t)$. Clearly,

$$\begin{aligned} \mathbf{P}(M^\mu(t) \in \hat{D}_1 \quad \forall t \in [s, s+T]) &= \mathbf{P}(B^\mu(s + [M^\mu]_t) \in \hat{D}_1 \quad \forall t \in [s, s+T]) \\ &\leq \mathbf{P}(B^\mu(r) \in \hat{D}_1 \quad \forall r \in [s, s + [M^\mu]_{s+T}]). \end{aligned} \quad (4.9)$$

By (4.8), $[M^\mu]_{s+T} \geq \delta T$ a.s. for all x, s . Hence

$$\mathbf{P}(B^\mu(r) \in \hat{D}_1 \quad \forall r \in [s, s + [M^\mu]_{s+T}]) \leq \mathbf{P}(B^\mu(r) \in \hat{D}_1 \quad \forall r \in [s, s + \delta T]). \quad (4.10)$$

By (4.6)–(4.7) and (4.9)–(4.10), it follows that

$$\sup_\mu \mathbf{P}(\tau^{x,s} > s + T) \leq \nu \triangleq \sup_\mu \mathbf{P}(B^\mu(r) \in \hat{D}_1 \quad \forall r \in [s, s + \delta T]),$$

and $\nu = \nu(\mathcal{P}) \in (0, 1)$. This completes the proof of Lemma 4.4. \square

Let

$$\mathcal{A}^*v \triangleq \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (b_{ij}(x, t)v(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\widehat{f}_i(x, t)v(x) \right) + \widehat{\lambda}(x, t)v(x)$$

and

$$B_i^*v \triangleq - \sum_{k=1}^n \frac{\partial}{\partial x_k} (\beta_{ik}(x, t, \omega) v(x)) + \bar{\beta}_i(x, t, \omega) v(x), \quad i = 1, \dots, N.$$

Here b_{ij} , x_i , β_{ik} are the components of b , β_i , and x .

Let $\rho \in Z_s^0$, and let $p = p(x, t, \omega)$ be the solution of the problem

$$\begin{aligned} d_t p &= \mathcal{A}^* p dt + \sum_{i=1}^N B_i^* p dw_i(t), \quad t \geq s, \\ p|_{t=s} &= \rho, \quad p(x, t, \omega)|_{x \in \partial D} = 0. \end{aligned}$$

By Theorem 3.4.8 from Rozovskii (1990), this boundary value problem has a unique solution $p \in Y^1(s, T)$.

Introduce an operator $\mathcal{M}_s : Z_s^0 \rightarrow Y^1(s, T)$ such that $p = \mathcal{M}_s \rho$.

The following lemma represents an analog of the so-called "the second energy inequality", or "the second fundamental inequality" known for the deterministic parabolic equations (see, e.g., inequality (4.56) from Ladyzhenskaya (1985), Chapter III).

Lemma 4.5 [*Dokuchaev (2005)*] *Problem (4.1) has a unique solution $p \in Y^2$ for any $\rho \in Z_s^1$, and*

$$\|p\|_{Y(s, T)^2} \leq C \|\rho\|_{Z_s^1}, \quad (4.11)$$

where $C > 0$ does not depend on ρ . This C depends on \mathcal{P} and on the supremums of the derivatives in Condition 3.1.

By Theorem 4.2 from Dokuchaev (2010), we have that $\kappa p(\cdot, T) = \mathcal{Q}^* \rho$, i.e.,

$$(\rho, \mathcal{Q}\Phi)_{Z_0^0} = (\rho, \kappa v(\cdot, 0))_{Z_0^0} = (p(\cdot, T), \kappa v(\cdot, T))_{Z_T^0} = (\kappa p(\cdot, T), \Phi)_{Z_T^0} \quad (4.12)$$

for $v = \mathcal{L}_T \Phi$. (See also Lemma 6.1 from Dokuchaev (1991) and related results in Zhou (1992)).

Suppose that there exists $\Phi \in Z_T^0$ such that $\kappa v(\cdot, 0) = v(\cdot, T)$ for $v = \mathcal{L}_T \Phi$, i.e., $v(\cdot, 0) = \mathcal{Q}\Phi = \Phi$. Let us show that $\Phi = 0$ in this case.

Let us assume that $\kappa = 1$. Since $\mathcal{Q}\Phi \in Z_0^0$, it follows that $\Phi \in H^0 = Z_0^0$. Let $p = \mathcal{M}_s \rho$ and $\bar{p}(x, t, 0) = \mathbf{E}p(x, t, \omega)$ (meaning the projection from Z_T^0 on $H^0 = Z_0^0$). Introduce an operator $\mathbf{Q} : H^0 \rightarrow H^0$ such that $\kappa \bar{p}(\cdot, T) = \mathbf{Q}\rho$. By (4.12), the properties of Φ lead to the equality

$$(\rho - \kappa p(\cdot, T), \Phi(\cdot, T))_{Z_T^0} = (\rho - \kappa \bar{p}(\cdot, T), \Phi(\cdot, T))_{H_0} = 0 \quad \forall \rho \in H^0. \quad (4.13)$$

It suffices to show that the set $\{\rho - \kappa\bar{p}(\cdot, T)\}_{\rho \in H^0}$ is dense in H^0 . For this, it suffices to show that the equation $\rho - \mathbf{Q}\rho = z$ is solvable in H^0 for any $z \in H^0$.

Let us show that the operator $\mathbf{Q} : H^0 \rightarrow H^0$ is compact. Let p be the solution of (4.11). This means that $\kappa \mathbf{E}p(\cdot, T) = \mathbf{Q}\rho$. By Lemma 4.2, it follows that

$$\|p(\cdot, \tau)\|_{Z_\tau^1} \leq C \|p(\cdot, s)\|_{Z_s^1}, \quad \tau \in [s, T], \quad (4.14)$$

where $C_* > 0$ is a constant that does not depend on p , s , and τ .

We have that $p|_{t \in [s, T]} = \mathcal{M}L_s p(\cdot, s)$ for all $s \in [0, T]$, and, for $\tau > 0$,

$$\begin{aligned} \|\bar{p}(\cdot, T)\|_{W_2^1(D)}^2 &\leq C_0 \|p(\cdot, T)\|_{Z_T^1}^2 \leq C_1 \inf_{t \in [0, T]} \|p(\cdot, t)\|_{Z_t^1}^2 \\ &\leq \frac{C_1}{T} \int_0^T \|p(\cdot, t)\|_{Z_t^1}^2 dt \leq \frac{C_2}{T} \|p\|_{X^1}^2 \leq \frac{C_3}{T} \|\Phi\|_{H^0}^2 \end{aligned}$$

for constants $C_i > 0$ that do not depend on Φ . Hence the operator $\mathbf{Q} : H^0 \rightarrow H^1$ is continuous. The embedding of H^1 into H^0 is a compact operator (see, e.g., Theorem 7.3 from Ladyzhenskaia (1985), Chapter I).

Let us show that if

$$\kappa\bar{p}(\cdot, T) = \kappa \mathbf{E}p(\cdot, T) = \mathbf{Q}\rho = p(\cdot, 0) \quad (4.15)$$

for some $\rho \in H^0$ then $\rho = 0$.

Let $\rho \in H^0$ be such that $\rho \geq 0$ a.e. and $\int_D \rho(x) dx = 1$. Let $a \in L_2(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^n)$ be independent from the process $(w(\cdot), \hat{w}(\cdot))$ such that $a \in D$ a.s. and it has the probability density function ρ . Let $p = \mathcal{M}_0 \rho$, and let $y(t) = y^{a,0}(t)$ be the solution of Ito equation (4.5) with the initial condition $y(0) = a$. By Theorem 6.1 from Dokuchaev (2011),

$$\int_D p(x, T, \omega) \Psi(x, \omega) dx = \mathbf{E} \left\{ \Psi(y^{a,s}(T)) \mathbb{I}_{\{\tau^{a,s} \geq T\}} \mid \mathcal{F}_T \right\} \quad \text{a.s.} \quad (4.16)$$

for all bounded functions $\Psi \in Z_T^0$. If $D = \mathbf{R}^n$ and $\mathbb{I}_{\tau^{a,s} \leq T} \equiv 1$, then this equality follows from Theorem 5.3.1 from Rozovskii (2001).

Equality (4.16) means that $p(x, T, \omega)$ is the conditional (given \mathcal{F}_T) probability density function of the vector $y(T)$ if the process $y(t)$ is killed at ∂D and if it is killed inside D with the rate of killing $\hat{\lambda}$. In particular, it follows that $p(x, t, \omega) \geq 0$ a.e. and

$$\mathbf{E} \int_D p(x, T, \omega) dx = \mathbf{E} \mathbb{I}_{\{\tau^{a,0} \geq T\}} = \mathbf{P}(\tau^{a,0} \geq T).$$

By Lemma 4.4, it follows that

$$\mathbf{E} \int_D p(x, T, \omega) dx = \int_D \mathbf{E} p(x, T, \omega) dx \leq \nu < 1.$$

By the linearity of problem (4.11), it follows that $p(x, T, \omega) \geq 0$ a.e. and

$$\int_D \mathbf{E}p(x, T, \omega)dx \leq \nu \int_D \rho(x)dx \quad (4.17)$$

for every non-negative $\rho(x)$.

Suppose that (4.15) holds for $\rho \in H^0$. Let

$$\rho_+(x) \triangleq \max(0, \rho(x)), \quad \rho_-(x) \triangleq \max(0, -\rho(x)).$$

Let p_+ and p_- be the solutions of (4.11) with $s = 0$ and with ρ replaced by ρ_{\pm} respectively. Let $\bar{p}_{\pm}(x, t) = \mathbf{E}p_{\pm}(x, t, \omega)$. By the definitions,

$$\bar{p}_+(\cdot, T) \triangleq \mathbf{Q}\rho_+, \quad \bar{p}_-(\cdot, T) \triangleq \mathbf{Q}\rho_-.$$

By (4.16), it follows that $\bar{p}_{\pm}(x, T) \geq 0$ for a.e. x . By (4.17), it follows that

$$\int_D \bar{p}_{\pm}(x, T)dx \leq \nu \int_D \rho_{\pm}(x)dx. \quad (4.18)$$

Let us assume first that $\rho_+ \neq 0$ and that $\kappa \in [0, 1]$. It follows that there exist a measurable set $D_0 \subset D$ such that $\text{mes}(D_0) > 0$ and that $\rho(x) > 0$ and $\bar{p}_+(x, T) < \rho(x)$ for all $x \in D_0$. It follows that $\bar{p}(x, T) = p_+(x, T) - p_-(x, T) < \rho(x)$. Therefore, $\kappa\bar{p}(x, T) \neq \rho(x)$ for $x \in D_0$. Hence $\kappa\bar{p}(\cdot, T) \neq \rho$ in this case. Similarly, we can show that $\kappa\bar{p}(\cdot, T) \neq \rho$ if $\rho_- \neq 0$ and $\kappa \in [0, 1]$.

Further, let us assume that $\kappa \in [-1, 0]$. Let $D_+ = \{x : \rho(x) \geq 0\}$, $D_- = \{x : \rho(x) < 0\}$. By the assumptions,

$$\int_{D_+} \rho(x)dx = \kappa \int_{D_+} \bar{p}(x, T)dx > 0, \quad \int_{D_-} \rho(x)dx = \kappa \int_{D_-} \bar{p}(x, T)dx < 0. \quad (4.19)$$

We have that

$$0 \leq - \int_{D_+} \bar{p}(x, T)dx \leq -\nu \int_{D_-} \rho(x)dx, \quad 0 \leq \int_{D_-} \bar{p}(x, T)dx \leq \nu \int_{D_+} \rho(x)dx.$$

Hence

$$- \int_{D_+} \bar{p}(x, T)dx \leq -\nu\kappa \int_{D_-} \bar{p}(x, T)dx, \quad \int_{D_-} \bar{p}(x, T)dx \leq \nu\kappa \int_{D_+} \bar{p}(x, T)dx.$$

Hence

$$\int_{D_+} \bar{p}(x, T)dx \geq \nu\kappa \int_{D_-} \bar{p}(x, T)dx, \quad \int_{D_-} \bar{p}(x, T)dx \leq \nu\kappa \int_{D_+} \bar{p}(x, T)dx. \quad (4.20)$$

The system of all inequalities in (4.19) and (4.20) can be satisfied only if all integrals here are zero. This means that $\rho = 0$.

We have proved that if (4.15) holds for $\rho \in H^0$ then $\rho = 0$. We had proved also that the operator \mathbf{Q} is compact. By the Fredholm Theorem, it follows that the equation $\rho - \mathbf{Q}\rho = z$ is solvable in H^0 for any $z \in H^0$. By (4.13), it follows that $\Phi = 0$. Therefore, the condition $\kappa u(\cdot, 0) = u(\cdot, T)$ fails to be satisfied for $u \neq 0$, $\varphi = 0$. Thus, $u = 0$ is the unique solution of problem (2.1)-(2.3) for $\xi = 0$ and $\varphi = 0$. Then the proof of Theorem 3.5 follows from Theorem 3.3. \square

Acknowledgment

This work was supported by ARC grant of Australia DP120100928 to the author.

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